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## ON SYMMETRIC FUNCTIONS.


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### H. APPLICATION OF THE THEORY TO COMPUTATION.

The limits of the present article do not permit of thoroughgoing application in the calculation of coefficients of terms of symmetric functions. The writer has used the present theory and the formula of G, 2, in such calculations, and by means of the parallel theory for the resultant, and the equivalent formula of A, 4, he has calculated all the coefficients of the normal forms of all resultants up to and including the resultant $R_{5,\,4}$.* By means of these methods the calculation is very easy. The advantages of the theory and formulas here developed in the calculation of tables of symmetric functions may be stated to be as follows: Not only is the symmetry of the table established by the fundamental relations, whereby half the coefficients are repeated, but also by the same relations the numerical equality of certain other coefficients in the same table and of others in different tables is immediately established. In addition to this many coefficients of a table which are not normal forms are easily reduced to such as are normal forms of a lower table and have been previously calculated. Also the coefficients of the completely reducible forms, which have the form of the general formula obtained for them in F, 2, are the values of $(-1)^w$. To this may be added certain coefficients whose value is zero by the fundamental conditions. [We might

*The results are published in the before-mentioned thesis.

also add the general formula, for the normal forms where $n=2$, which can be proved without difficulty, viz :

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & 2^{\lambda_2} \\ 0^m & m & 0 \end{pmatrix} = (-1)^{\lambda_1+\lambda_2} \frac{m(\lambda_1+\lambda_2-1)!}{\lambda_1!\,\lambda_2!}].$$

When all these means of obtaining coefficients have been applied, the actual number of normal forms in a table requiring calculation is comparatively very small, and by means of the formula of F, 2, the calculation is easily made, giving the coefficient at once as the sum of earlier calculated coefficients.

## NOTE ON ELIMINATION BY MEANS OF SYMMETRIC FUNCTIONS.

In behalf of the extension of the method given under I, B, the following details may be added :

1. On the Resolution of Aronhold's Operator into Three Operators.

The term containing $(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$ in $\delta R_{m,n}$ must have come from $a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m} \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid$ by the use of $b_iD_{a_i}$ in as many ways as there are operators of this kind when $i$ takes all values from $i=0$, to $i=n$. $b_iD_{a_i}$ applied to $a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m} \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid$ gives

$$(\lambda_i+1)b_i \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid (a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$$

$$=(\lambda_i+1)b_i \mid 0^{\lambda_0}1^{\lambda_1}\ldots.i^{\lambda_i+1}\ldots.m^{\lambda_m} \mid (a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m},$$

and the coefficient of $(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$ in $\delta R_{m,n}$ is

$$\sum_{i=0}^{i=n} (\lambda_i+1)b_i \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid \equiv 0.$$

Thus Aronhold's operator is resolved into three operators :

(1). $0, 1, 2, \ldots.n$ *applied to* $\mid 0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid$ .

(2). *The literal operators* $b_0, b_1, \ldots b_n$ *applied to the preceding.*

(3). *The numerical operators* $\lambda_0+1, \lambda_1+1, \ldots.\lambda_n+n$.

Here $\lambda_i+1$ associated with $b_i$ is the exponent of $i$ in the associated stroked form which results from the first operation. Of course $\lambda_0+\lambda_1+\ldots.\lambda_m=n-1$, and there will be

$$\frac{m(m+1)\ldots.(m+n-1)}{1.2\ldots.(n-1)}=e,$$

(the number of homogeneous products of $m$ elements to $n-1$ dimensions) such identical equations of the form

$$\sum_{i=0}^{i=n} (\lambda_i+1)b_i \mid i0^{\lambda_0}1^{\lambda_1}\ldots\ldots m^{\lambda_m} \mid \equiv 0.$$

The reader will observe the identity of this equation with the one from which the recurrence formula for the normal forms of symmetric functions was derived.

## 2. On the Number of Functions Requiring Calculation in $R_{m,n}$, and the Sufficiency of Aronhold's Operator.

It is evident that every symmetric function of the form

$$b_0{}^m \Sigma(\beta_1)^{\kappa_1}(\beta_2)^{\kappa_2}\ldots\ldots(\beta_n)^{\kappa_n},$$

in $R_{m,n}$, where $\kappa_1 \geqq \kappa_2 \geqq \ldots\ldots \geqq \kappa_n > 0$, can be reduced to the form

$$(-1)^{n\kappa_n}(b_n)^{\kappa_n}(b_0)^{m-\kappa_n} \Sigma (\beta_1)^{\kappa_1-\kappa_n}(\beta_2)^{\kappa_2-\kappa_n}\ldots\ldots(\beta_{n-1})^{\kappa_{n-1}-\kappa_n}.$$

Hence only those functions require calculation which contain in any term less than all the $n$ roots; and of these we know all the fundamental symmetric functions,

$$b_0\Sigma\beta_1, \quad b_0\Sigma\beta_1\beta_2, \quad \ldots\ldots \quad b_0\Sigma\beta_1\beta_2\ldots\ldots\beta_{n-1}.$$

Let $N$ denote the number of functions requiring calculation. All the terms in $R_{m,n}$ in which $a_m$ is not a factor, contain functions in which $n$ roots enter, and are therefore of the before mentioned reducible form, and are dependent upon such forms as do not contain $n$ roots in a term. These latter irreducible (in this sense) forms are all found among the terms of $R_{m,n}$ which contain a power of $a_m$, and among these are none which contain $n$ roots in a term. We may note that the terms of the form $(b_0)^m(a_m)^{m-r}(a_{m-1})^r\Sigma\beta_1\beta_2\ldots\ldots\beta_r$ from $r=0$, to $r=n-1$, $n$ in number are found among them and are known. We have $N+n=$the number of terms in $R_{m,n}$ containing $a_m$. The whole number of terms or functions, $f$, in the resultant $R_{m,n}$ is

$$f=\frac{(m+1)(m+2)\ldots\ldots(m+n)}{1.2\ldots\ldots n}.$$

The number of them not containing $a_m$ could be found by putting $a_m=0$. It is the same as the number of terms in $R_{m-1,n}$, and is

$$\frac{m(m+1)\ldots\ldots(m+n-1)}{1.2\ldots\ldots n}.$$

Therefore,

$$N+n=\frac{(m+1)(m+2)\ldots\ldots(m+n)}{1.2\ldots\ldots n}-\frac{m(m+1)\ldots\ldots(m+n-1)}{1.2\ldots\ldots n}=f-\frac{m}{n}e$$

$$=\frac{(m+1)(m+2)\ldots.(m+n-1)}{1.2\ldots.(n-1)}=e, \text{ or}$$

$$N+n=f-\frac{m}{n}e=e, \text{ and } N+n=e=\frac{n}{m+n}f\leqq\tfrac{1}{2}f,$$

and we see:

(1). *The number of identical linear equations furnished by Aronhold's operator*, and containing the $N$ unknown functions together with $n$ others, *is just equal to the number $N+n$, and therefore just sufficient to compute both the unknown $N$ as well as also the $n$ other functions*, if we regard the latter as unknown. This proves the sufficiency of Aronhold's operator for calculating the resultant by means of symmetric functions.

(2). *Certainly less than one-half of the whole number of symmetric functions which enter into the resultant require calculation.*

If $m=n$, $N+n=\frac{1}{2}f$,
$n=1$, $N=0$, $e=1$, $f=2$,
$n=2$, $N=1$, $e=3$, $f=6$,
$n=3$, $N=7$, $e=10$, $f=20$, etc.,

which agrees with the fact that we found 7 functions requiring calculation in $R_{3,3}$.

### 3. Farther Remarks on Calculating $R_{m,n}$.

(1). *Recurrence methods.*

Since the sum of all the terms which do not contain $a_m$ is equal to $b_n R_{m-1,n}$, we have

$$R_{m,n}=b_n R_{m-1,n}+\Sigma\lambda\sum_{i=0}^{i=n}a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}\mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid$$

where the exponent of $a_m$ must not be zero, and where $\lambda_0+\lambda_1+\ldots.\lambda_m=n-1$. The second portion of the right hand expression contains all the unknown functions while the first term is a previously calculated resultant. By giving $m$ and $n$ special values, or requiring them to satisfy certain relations, like $m=n+1$, etc., various recurrence formulas may be obtained and used.

(2). *Direct calculation of $R_{m,n}$.*

If one does not choose to proceed by recurrence formulas we have shown that Aronhold's operator furnishes a sufficient number of equations that are written down by an easy rule, to calculate $R_{m,m}$ directly and independently. From $R_{m,m}$ we can obtain $R_{m,n}$ where $n=m-r$, at once by the formula

$$R_{m,m-r}=(-1)^{mr}\left(\frac{R_{m,m}}{a_m{}^r}\right)b_m=b_{m-1}=\ldots.b_{m-r+1}=0.$$

To this may be added the relation, $R_{m,n}=(-1)^{mn}R_{n,m}$.

## BIBLIOGRAPHICAL REFERENCES.

In the preparation of this paper many of the fundamentals of Algebra have been assumed as being familiar. For the benefit of any readers who may feel the need of them, the following references are given to cover the ground which has thus been assumed. The references are intended in no sense to form a complete bibliography.

In the theorems concerning order and weight of symmetric functions, see Faà di Bruno, Einleitung in die Theorie der Binären Formen, German Translation by Dr. Theodor Walter, Leipzig, B. G. Teubner, 1881. s. TU. 13; also Burnside and Panton, Theory of Equations, second edition, London: Longmans, Green & Co., 1886, pp. 299 *et seq.*

On corresponding and conjugate forms of symmetric functions, see Faà di Bruno, loc. cit. Anhang, s. 302 *et seq.*

For tables of symmetric functions. Ibid. s. 311 *et seq.*

For a pretty complete bibliography of the literature up to 1881, of symmetric functions, resultants, and related subjects, see Faà di Bruno, Anhang. s. 373 *et seq.*

On the resultant of two binary forms and its elementary properties, see Faà di Bruno, loc. cit. §5. Ueber die Bildung der Resultanten. s. 50 *et seq.* §6. Eigenschaften der Resultanten, s. 75 *et seq*; also Gordan, Invariantentheorie, Erster Band, Leipzig, B. G. Teubner, 1885. §§ 10 u. 11, s. 145 *et seq*; also Weber, Lehrbuch der Algebra, Zweite Auflage, Erster Band, Braunschweig, F. Vieweg und Sohn, 1898. §53, Resultanten, s. 175 *et seq*; also Burnside and Panton, loc. cit. Chapter XIII, pp. 318 *et seq.*

On the resultant in terms of symmetric functions, see the references already given.

In addition to the preceding, the works of Salmon either in German or in English may be consulted.

On substitutions and their application to determinants, so far as employed in this paper, see Gordan, loc. cit., s. 1—31.

On Aronhold's operator, see Gordan, Invariantentheorie, Zweiter Band, Leipzig, B. G. Teubner, 1887. s. 60 *et seq.*

For proof of the proposition that Aronhold's operator applied to the resultant gives identically zero, or that $\delta R_{m,n} \equiv 0$, see Faà di Bruno, loc. cit. s. 79, Anmerkung 3.

On corresponding matrices and corresponding determinants, see Gordan, Invariantentheorie, Erster Band, s. 94 *et seq.*

On the number of homogeneous products of $n$ elements to $r$ dimensions, see Todhunter, Algebra, London: Macmillan & Co., 1881. p. 316.